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ANALYTICAL AND NUMERICAL MODELING OF WATER WAVES

ABSTRACT

of the PhD thesis for the degree of
doctor of Philosophy (PhD) in the specialty
«6D070500-Mathematical and computer modeling»

The relevance of the research topic. The theory of surface gravity waves is considered one of the classical branches of hydrodynamics. However, modeling wave breaking remains a complex problem. To calculate wave breaking, a seminal method was proposed in the 1979 work of S. Hibberd and D. Peregrine, based on a conservative numerical solution of the nonlinear shallow water equations. Although this method yielded physically realistic results, a 1989 study by C. Synolakis demonstrated its diminished reliability during the wave run-up phase. Furthermore, it is important to consider that a significant amount of energy is dissipated after the wave breaks. Consequently, later versions of the algorithm, such as the one discussed in the 1981 work by A. Packwood and D. Peregrine, incorporated artificial dissipative terms like viscosity and friction to more accurately describe the wave amplitude. Instead of introducing such artificial terms, the model considered in this dissertation describes the phenomenon of wave breaking by introducing a new, physically-grounded variable — enstrophy (the square of vorticity) drawing upon the works of G. Richard and S. Gavrilyuk from 2012 and 2015. However, unlike the aforementioned studies, the generation of enstrophy in this model is associated with turbulent dissipation rather than the formation of a shock wave.

Describing the dynamics of water waves in a medium with a moving boundary is a prominent issue in the contemporary fields of applied mathematics, physics, and marine engineering. The displacement of the boundary, resulting from changes in the coastline, the influence of hydraulic structures, or other natural and anthropogenic factors, complicates the propagation properties of waves. Although problems with moving boundaries are encountered in many scientific disciplines, their significance is particularly pronounced in the modeling of wave processes. In this area, the Korteweg-de Vries (KdV) and Kawahara equations, which describe nonlinear and dispersive effects, serve as fundamental tools. They are widely used, for instance, to model the propagation of gravity waves in shallow water (such as tsunamis and waves in canals), ion-acoustic waves in plasma, and processes in nonlinear optics. Such research contributes to achieving significant practical results in coastal protection, navigational safety, and the design of energy facilities. Therefore, the study and adaptation of the Kawahara and Korteweg-de Vries equations for environments with moving boundaries represent a vital scientific

direction aimed at understanding the complex nature of water waves, predicting their behavior, and managing their effects.

In conclusion, the mathematical modeling of nonlinear wave processes is one of the rapidly developing fundamental directions in modern mathematical physics and computational sciences. The high relevance of the chosen research topic is evidenced by the growing scientific interest in this field, especially in studies based on complex mathematical models and contemporary methods. A clear indicator of this is the steady increase in the number of scientific publications dedicated to this topic in leading international databases such as Web of Science, Scopus, and MathSciNet.

The aim of the PhD thesis is to investigate analytical and numerical solutions of the traveling wave type for modeling wave breaking phenomena in shallow water.

A further objective is to analyze the propagation patterns of water waves in a domain with a moving boundary and to study methods for finding analytical and numerical solutions to the Korteweg-de Vries and Kawahara equations used to describe these phenomena.

To achieve the aim of the dissertation, the main objectives of the following research are considered:

- To find analytical and numerical solutions of the traveling wave type for modeling wave breaking phenomena in shallow water;
- To find a numerical solution for the Korteweg-de Vries equation in a domain with a moving boundary;
- To investigate the existence of a global solution for the Kawahara equation in a domain with a moving boundary;
- To find a numerical solution for the Kawahara equation in a domain with a moving boundary.

Object of the PhD thesis. A nonlinear mathematical model of coastal waves, as well as the Korteweg-de Vries and Kawahara equations in a domain with a moving boundary.

The methods of scientific research. Various methods from the theory of partial differential equations, functional analysis, and nonlinear analysis were employed in solving the problems of this dissertation. Specifically:

To prove the existence of a global solution:

- Formulating the equations in a weak sense;
- The Galerkin method;
- Obtaining a priori estimates;
- Convergence of a weakly bounded sequence of approximate solutions;
- Proving that the global solution can be extended for all time.

To study the numerical solution:Method of fundamental solutions and method of Green's functions;

- The Galerkin method;

- The finite difference method;
- The finite element method.

Scientific novelty of the work. The nonlinear model of coastal waves and the investigation of analytical and numerical solutions for the Korteweg-de Vries and Kawahara equations in domains with moving boundaries, as studied in this dissertation, represent new problems. The issues under consideration have been largely uninvestigated or have only been demonstrated for special cases. Therefore, this research aims to extend previously known results and to obtain new findings.

Theoretical and practical significance of the results. The research work is primarily theoretical and fundamental, and its scientific significance is associated with the application of profound, modern results from the theory of hydrodynamics and mathematical physics, as well as the creation of new, original methods for investigation and analysis.

Publications. On the topic of the thesis 7 papers were published, including 2 publications in a high-ranking scientific journal, indexed in the Web of Science and Scopus, 3 publications in scientific journals included in the list recommended by the Committee on the Control of Education and Science of the MES RK for publication of the main scientific results of scientific activities, 2 publications in materials of foreign international conferences**.**

The results on the topic of the thesis were published in the following papers:

Publication in the high-ranking scientific journals

1 N. Koshkarbayev. Travelling breaking waves // Bulletin of the South Ural State University. Ser. Mathematical Modelling, Programming & Computer Software, 2023, - Vol. 16, No. 2, - P. 49–58. DOI: 10.14529/mmp230205. Scopus SJR=0.245(Q3), CiteScore=0,9, Scopus Percentile=25.

2 N. Koshkarbayev. Blowing-up solutions of the shallow water equations // Journal of Nonlinear Modeling and Analysis, 2025, - Vol. 7, No. 4, -P. 1523-1531., DOI:10.12150/jnma.2025.1523. Scopus SJR=0.322(Q2), CiteScore=1,5, Scopus Percentile=56.

CCES

1 N. Koshkarbayev, B. T. Torebek. Nonexistence of travelling wave solution of the Korteweg-de Vries Benjamin Bona Mahony equation // International Journal of Mathematics and Physics, 2019, - Vol. 10, No. 1 – P. 51-55.

2 N. Koshkarbayev, B. T. Torebek. Blowing-up solutions of the shallow water equations // Kazakh Mathematical Journal. - 2019, - Vol. 19, No. 2, pp. 70–77.

3 N. Koshkarbayev, B. T. Torebek. About a singular travelling wave solution of the KDV-BBM equation // KazNITU Bulletin. 2019, -Vol 6, pp. 644-649.

Publications in materials of international conferences

1 N. Koshkarbayev, B. T. Torebek. On a mathematical model of breaking travelling waves // All Russian Conference and School for Young Scientists Dedicated to the 100th Anniversary of Academician L.V. Ovsyannikov. "Mathematical Problems of Continuous Media Mechanics" May 13-17, - 2019. - P. 263.

2 N. Koshkarbayev, Kawahara equation in domains with moving boundaries // Traditional International April scientific conference in honor of the Science Day. Almaty, April 10, - 2025. - P. 224-225.

The structure and scope of the thesis. The PhD thesis includes a title page, content, introduction, four chapters, conclusion and list of references, consisting of 62 titles. The total volume of the thesis is 112 pages.

The main content of the thesis.

In the first chapter, we consider the following system for a model of breaking waves in shallow water:

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial hU}{\partial x} = 0, \\ \dfrac{\partial hU}{\partial t} + \dfrac{\partial}{\partial x}\left(hU^2 + \dfrac{gh^2}{2} + h^3\varphi + \dfrac{h^2\ddot{h}}{3} \right) = \dfrac{\partial}{\partial x}\left(\dfrac{4}{\mathrm{Re}} h^3 \sqrt{\varphi} \dfrac{\partial U}{\partial x} \right), \\ \dfrac{\partial h\varphi}{\partial t} + \dfrac{\partial hU\varphi}{\partial x} = \dfrac{8h\sqrt{\varphi}}{\mathrm{Re}} \left(\dfrac{\partial U}{\partial x} \right)^2 - C_r h\varphi^{3/2} \end{cases} \tag{0.1}$$

where h is the water depth, U is the depth-averaged horizontal velocity, g is the gravitational constant and φ is the enstrophy variable. The system depends on two parameters: Re – the Reynolds number and C_r – the turbulent dissipation parameter. In system (0.1), the first equation represents the conservation of mass, the second is the conservation of momentum, and the third is the enstrophy equation. The investigation focuses on finding a traveling wave solution for this system, proving the existence of such a solution, determining the critical value of Re_{cr} and justifying its physical and engineering consequences.

In the second chapter, we consider the following nonlinear initial-boundary value problem based on the classical Korteweg-de Vries equation:

$$\begin{cases} \vartheta_\tau + \vartheta\vartheta_\xi + \vartheta_{\xi\xi\xi} = 0, \quad (\xi,\tau) \in Q_\tau, \\ \vartheta(\alpha(\tau),\tau) = \vartheta(\beta(\tau),\tau) = 0, \\ \vartheta_\xi(\beta(\tau),\tau) = 0, \qquad \tau \in [0,T], \\ \vartheta(\xi,0) = \vartheta_0(\xi), \qquad \xi \in D_0 \end{cases} \tag{0.2}$$

where $Q_\tau = \{(\xi,\tau)\in R^2 \mid \xi \in D_\tau, \tau\in(0,T), T>0\}$ is a domain with a moving boundary and its boundary is $D_\tau = \{\xi \in R \mid \alpha(\tau)<\xi<\beta(\tau), \tau>0\}$. The research involves proving the existence of a solution for system (0.2) and finding its numerical solution.

In the third chapter, we consider the following nonlinear initial-boundary value problem based on the classical Kawahara equation:

$$\begin{cases} \vartheta_\tau + \vartheta\vartheta_\xi + \vartheta_{\xi\xi\xi} - \vartheta_{\xi\xi\xi\xi\xi} = 0, \quad (\xi,\tau)\in Q_\tau, \\ \vartheta(\alpha(\tau),\tau)=\phi_1, \ \vartheta(\beta(\tau),\tau)=\phi_2, \\ \vartheta_\xi(\alpha(\tau),\tau)=\phi_3, \ \vartheta_\xi(\beta(\tau),\tau)=\phi_4, \\ \vartheta_{\xi\xi}(\beta(\tau),\tau)=\phi_5, \qquad \tau\in[0,T], \\ \vartheta(\xi,0)=\vartheta_0(\xi), \qquad \xi\in D_0. \end{cases} \tag{0.3}$$

The investigation involves proving the existence of a solution for system (0.3) and finding a numerical solution for the problem defined by system (0.3) with homogeneous boundary conditions.